

A VARIANT OF THE MUKAI PAIRING VIA DEFORMATION QUANTIZATION.

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ABSTRACT. We give a new method to prove a formula computing a variant of Caldararu's Mukai pairing [Cal1]. Our method is based on some important results in the area of deformation quantization. In particular, part of the work of Kashiwara and Schapira in [KS] as well as an algebraic index theorem of Bressler, Nest and Tsygan in [BNT],[BNT1] and [BNT2] are used. It is hoped that our method is useful for generalization to settings involving certain singular varieties.

1. INTRODUCTION.

Let X denote a smooth proper complex variety (we remind the reader that X has the Zariski topology). We denote the corresponding (compact) complex manifold by X^{an} . In [Cal1], A. Caldararu defined a Mukai pairing $\langle -, - \rangle_M$ on $\mathrm{HH}_\bullet(X)$, the Hochschild homology of X . On the other hand, one has the Hochschild-Kostant-Rosenberg (HKR) isomorphism $I_{HKR} : \mathrm{HH}_\bullet(X) \rightarrow \oplus_i H^{i-\bullet}(X, \Omega_X^i)$. It was implicitly proven in [Mar1] (and explicitly so in [Ram1] following [Mar1]) that

$$\langle a, b \rangle_M = \int_X I_{HKR}(b) \wedge J(I_{HKR}(a)) \wedge \mathrm{td}(T_X)$$

where J is the involution multiplying an element of $H^{i-\bullet}(X, \Omega_X^i)$ by $(-1)^i$. This result has recently been of interest: applications of this and related results appear, for instance, in [HMS],[MaS] and [Ram4]. A closely related pairing was $\langle -, - \rangle_{Shk}$ was constructed in [Ram3] following D. Shklyarov in [Shkl]. It turns out that the latter pairing is directly related to a natural definition of Fourier-Mukai transforms in Hochschild homology (see [Shkl] and [Ram3]). This definition of Fourier-Mukai transforms in Hochschild homology is equivalent to an earlier, but less direct definition in [Cal1] (also see Section 4.3 of [KS]). A careful comparison between this pairing and Caldararu's Mukai pairing was performed in [Ram3] to show that

Theorem 1.

$$(1) \quad \langle a, b \rangle_{Shk} = \int_X I_{HKR}(a) \wedge I_{HKR}(b) \wedge \mathrm{td}(T_X).$$

In these notes, we provide a different proof of this result based on the work of Kashiwara-Schapira [KS] and an algebraic index theorem of Bressler, Nest and Tsygan in [BNT],[BNT1] and [BNT2] (the latter being a very important result in the general area of deformation quantization). Unlike the earlier approach from [Mar1], [Ram1] and [Ram3] (also see [Ram5] for further details), this approach requires that we work over \mathbb{C} . However, it gives a clear connection (hitherto missing) between the computation of a ‘‘Mukai pairing’’ and a large body of work in deformation quantization, algebraic index theorems and related topics. We also point out that essentially the same result has been proven in [Griv] using what we use from [KS] and a deformation to the normal cone argument. While the (interesting) approach in [Griv] is far more concise than the one via [Mar1], [Ram1] and [Ram3], the argument there is geometric and not intrinsic to X . Readers with some background in deformation quantization and algebraic index theory would also find the approach in this note far more concise than the earlier one (that in [Mar1], [Ram1] and [Ram3]), while remaining algebraic and intrinsic to X in nature. Further, unlike the earlier approach, this method is likely to lend itself to generalization to more general settings involving certain singular varieties. We also remark that as far as (the above cited as well as other) recent applications are concerned, a formula for $\langle -, - \rangle_{Shk}$ is as useful/suitable as one for $\langle -, - \rangle_M$.

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2. PRELIMINARIES.

Let $\omega_X := \Omega_X^n[n]$. Let $\Delta : X \rightarrow X \times X$ denote the diagonal embedding. Recall from [KS] that one has the following commutative diagram in the bounded derived category $D^b(\mathcal{O}_X)$ of coherent sheaves on X .

$$\begin{array}{ccc} \Delta^* \Delta_* \mathcal{O}_X & \xrightarrow{\text{td}} & \Delta^! \Delta_* \omega_X \\ \downarrow I_{HKR} & & \uparrow \widehat{I_{HKR}} \\ \oplus_i \Omega_X^i[i] & \xrightarrow{\tau} & \oplus_i \Omega_X^i[i] \end{array}$$

Let D denote the map on hypercohomology induced by $\text{td} : \Delta^* \Delta_* \mathcal{O}_X \rightarrow \Delta^! \Delta_* \omega_X$. Let $I_{HKR}, \widehat{I_{HKR}}$ and τ continue to denote the maps induced on hypercohomology by $I_{HKR}, \widehat{I_{HKR}}$ and τ respectively. Applying hypercohomologies, one obtains the following commutative diagram.

$$(2) \quad \begin{array}{ccc} \mathbb{H}^{-\bullet}(X, \Delta^* \Delta_* \mathcal{O}_X) & \xrightarrow{D} & \mathbb{H}^{-\bullet}(X, \Delta^! \Delta_* \omega_X) \\ \downarrow I_{HKR} & & \uparrow \widehat{I_{HKR}} \\ \oplus_i \mathbb{H}^{i-\bullet}(X, \Omega_X^i) & \xrightarrow{\tau} & \oplus_i \mathbb{H}^{i-\bullet}(X, \Omega_X^i) \end{array}$$

The map $\widehat{I_{HKR}}$ has been constructed in [KS], Section 5.4¹. M. Kashiwara and P. Schapira show us in [KS]² that

Proposition 1. Theorem 1 is equivalent to the assertion that the map τ in (2) is the wedge product with $\text{Td}(TX)$.

Proof. Let X, Y be smooth projective varieties over \mathbb{C} . Recall that any $\Phi \in D_{coh}^b(X \times Y)$ gives an integral transform $\Phi_*^{cal} : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(Y)$ (see Section 4.3 of [Cal1]). On hypercohomologies, Corollary 4.2.2 of [KS] yields a pairing

$$\langle -, - \rangle_{KS} : \text{HH}_\bullet(X) \otimes \text{HH}_\bullet(X) \rightarrow \mathbb{C}.$$

We remark that $\text{HH}_\bullet(X)$ is also the hypercohomology of the complex of Hochschild chains of \mathcal{O}_X^{op} , which is equal to $\text{HH}_\bullet(X)$ since $\mathcal{O}_X^{op} = \mathcal{O}_X$. In particular, we are not making this identification via the duality map described at the end of Section 4.1 of [KS]. Lemma 4.3.4 of [KS] then tells us that after identifying $HH_\bullet(X \times Y)$ with $HH_\bullet(Y) \otimes HH_\bullet(X)$,³

$$(3) \quad \Phi_*^{cal}(\alpha) = \langle \text{Ch}(\Phi), \alpha \rangle_{KS}.$$

Let $\Phi = \mathcal{O}_\Delta$ (Δ here denoting the diagonal in $X \times X$). In this case, $\Phi_*^{cal} = \text{id}$ (see Section 5 of [Cal1]). Then, by Theorem 5 of [Ram3]⁴, $\text{Ch}(\Phi) = \sum_i e_i \otimes f_i$ where the e_i and f_i are homogenous bases of $\text{HH}_\bullet(X)$ such that $\langle f_j, e_i \rangle_{Shk} = \delta_{ij}$. On the other hand, equation (3) applied to $\alpha = e_i$ tells us that $\langle f_j, e_i \rangle_{KS} = \delta_{ij}$, thus showing that $\langle -, - \rangle_{KS} = \langle -, - \rangle_{Shk}$. Finally, the end of Section 5.4 of [KS] shows us that

$$\langle a, b \rangle_{KS} = \int_X I_{HKR}(a) \wedge \tau(I_{HKR}(b)).$$

□

We therefore, need to show that $\tau = (- \wedge \text{Td}(TX))$. In our method, the following proposition from [KS], Chapter 5 is the first step in this direction.

Proposition 2. (i) $\Delta^* \Delta_* \mathcal{O}_X$ is a ring object in $D^b(\mathcal{O}_X)$, and $\Delta^! \Delta_* \omega_X$ is a left module object over $\Delta^* \Delta_* \mathcal{O}_X$ in $D^b(\mathcal{O}_X)$.

(ii) Further, td is a morphism of left $\Delta^* \Delta_* \mathcal{O}_X$ modules in $D^b(\mathcal{O}_X)$.

¹A similar map has been constructed in Section 1 of [Ram1].

²We remark that all constructions/results in Chapter 5 of [KS], which are done in the setting of complex manifolds, work in the algebraic setting that we are working in.

³ $X \times Y$ is viewed as $Y \times X$ while making this identification.

⁴Note that we are not using any part of [Ram3] that depends on the Mukai pairing formula computed in [Mar1] and [Ram1].

Proof. The ring structure of $\Delta^* \Delta_* \mathcal{O}_X$ in $D^b(\mathcal{O}_X)$ is given by the composite map

$$\Delta^* \Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \Delta^* \Delta_* \mathcal{O}_X \cong \Delta^*(\Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}}^{\mathbb{L}} \Delta_* \mathcal{O}_X) \xrightarrow{\Delta^* \mu} \Delta^* \Delta_* \mathcal{O}_X$$

where μ is induced by the product map $\Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}} \Delta_* \mathcal{O}_X$.

The module structure of $\Delta^! \Delta_* \omega_X$ over $\Delta^* \Delta_* \mathcal{O}_X$ is realized via the composite map

$$\Delta^* \Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \Delta^! \Delta_* \omega_X \cong \Delta^!(\Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}}^{\mathbb{L}} \Delta_* \omega_X) \xrightarrow{\Delta^! \nu} \Delta^! \Delta_* \omega_X.$$

Here, ν is the composite map

$$\Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_{X \times X}}^{\mathbb{L}} \Delta_* \omega_X \cong \Delta_*(\Delta^* \Delta_* \mathcal{O}_X \otimes_{\mathcal{O}_X} \omega_X) \rightarrow \Delta_* \omega_X$$

the last arrow being induced by the adjunction $\Delta^* \Delta_* \mathcal{O}_X \rightarrow \mathcal{O}_X$.

The morphism td was constructed in [KS] as follows.

$$\begin{aligned} \Delta^* \Delta_* \mathcal{O}_X &\cong \mathcal{O}_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \Delta^* \Delta_* \mathcal{O}_X \cong \Delta^!(\mathcal{O}_X \boxtimes \omega_X) \otimes_{\mathcal{O}_X}^{\mathbb{L}} \Delta^* \Delta_* \mathcal{O}_X \cong \Delta^!((\mathcal{O}_X \boxtimes \omega_X) \otimes_{\mathcal{O}_{X \times X}}^{\mathbb{L}} \Delta_* \mathcal{O}_X) \\ &\cong \Delta^!((\mathcal{O}_X \boxtimes \omega_X) \otimes_{\mathcal{O}_{X \times X}}^{\mathbb{L}} \Delta_* \mathcal{O}_X) \cong \Delta^! \Delta_* \omega_X \end{aligned}$$

That td is a morphism of left $\Delta^* \Delta_* \mathcal{O}_X$ -modules is more or less a direct consequence of the fact that $\otimes_{\mathcal{O}_X}^{\mathbb{L}}$ is associative. \square

Corollary 1. For all $\alpha \in \oplus_i H^{i-\bullet}(X, \Omega_X^i)$, $\tau(\alpha) = \alpha \wedge \tau(1)$.

Proof. The ring structure of $\Delta^* \Delta_* \mathcal{O}_X$ induces a product \bullet on $\mathbb{H}^{-\bullet}(X, \Delta^* \Delta_* \mathcal{O}_X)$. By Proposition 2,

$$D(a \bullet b) = a \bullet D(b)$$

for all $a, b \in \mathbb{H}^{-\bullet}(X, \Delta^* \Delta_* \mathcal{O}_X)$. It follows from Lemma 5.4.7 of [KS] that for all $a, b \in \mathbb{H}^{-\bullet}(X, \Delta^* \Delta_* \mathcal{O}_X)$,

$$\widehat{I_{HKR}}(I_{HKR}(a) \wedge \beta) = a \bullet \widehat{I_{HKR}}(\beta)$$

The desired corollary now follows from the fact that I_{HKR} and $\widehat{I_{HKR}}$ are isomorphisms. \square

Recall that for any $E \in D^b(\mathcal{O}_X)$, one has the *Chern character* $\text{ch}(E) \in \mathbb{H}^0(X, \Delta^* \Delta_* \mathcal{O}_X)$. By Theorem 4.5 of [Cal2], $I_{HKR}(\text{ch}(E))$ is the Chern character of E in the classical sense. The *Euler class* $\text{eu}(E)$ is defined as the element $\widehat{I_{HKR}}^{-1}(D(\text{ch}((E))))$ of $\oplus_i H^i(X, \Omega_X^i)$. Note $\tau(1) = \text{eu}(\mathcal{O}_X)$. In order to compute the $\langle -, - \rangle_{Shk}$, we therefore, need to show that

$$\text{eu}(\mathcal{O}_X) = \text{Td}(T_X).$$

Before we proceed, let us make a clarification. Recall that $\Delta^* \Delta_* \mathcal{O}_X$ is represented in the derived category $D^-(\mathcal{O}_X)$ of bounded above complexes of quasi-coherent sheaves on X by the complex of $\widehat{\mathcal{C}}_{\bullet}(\mathcal{O}_X)$ of completed Hochschild chains (after turning it into a cochain complex by inverting degrees). Recall from [Y] that $\widehat{\mathcal{C}}_n(\mathcal{O}_X) := \varprojlim_k \frac{\mathcal{O}_X^{\otimes n+1}}{I_n^k}$ where I_n is the kernel of the product map $\mathcal{O}_X^{\otimes n+1} \rightarrow \mathcal{O}_X$. Let $\mathcal{C}_{\bullet}(\mathcal{O}_X)$ be the complex of sheaves of X associated to the complex of presheaves $U \mapsto \mathcal{C}_{\bullet}(\Gamma(U, \mathcal{O}_X))$ (the Hochschild chain complex here being the naive algebraic one). One similarly defines $\mathcal{C}_{\bullet}^{\text{red}}(\mathcal{O}_X)$ using reduced Hochschild chains. There are natural maps $\mathcal{C}_{\bullet}^{\text{red}}(\mathcal{O}_X) \leftarrow \mathcal{C}_{\bullet}(\mathcal{O}_X) \rightarrow \widehat{\mathcal{C}}_{\bullet}(\mathcal{O}_X)$ of complexes of sheaves on X which are quasiisomorphisms. In the following section, when thinking of the complex of Hochschild chains on X , we shall be thinking of $\mathcal{C}_{\bullet}^{\text{red}}(\mathcal{O}_X)$ (which has the same hypercohomology as $\widehat{\mathcal{C}}_{\bullet}(\mathcal{O}_X)$).

3. THE EULER CLASS OF \mathcal{O}_X .

It remains to show that $\text{eu}(\mathcal{O}_X) = \text{Td}(T_X)$. The original intrinsic computation for this from [Mar1] (see [Ram1] for details) is very lengthy and involved. Further, its connections to deformation quantization and related areas are not clear. Another, more recent proof due to [Griv] uses deformation to the normal cone. We now sketch our new approach to this question. Let \mathcal{D}_X denote the sheaf of (algebraic) differential operators on X . Recall that the Hochschild-Kostant-Rosenberg quasiisomorphism on Hochschild chains induces an isomorphism $I_{HKR} : \text{HC}_0^{\text{per}}(\mathcal{O}_X) \rightarrow \prod_{p=-\infty}^{\infty} H^{2p}(X^{\text{an}}, \mathbb{C})$. On the other hand, a construction very similar to the trace density construction of Engeli-Felder on Hochschild chains induces an isomorphism $\chi : \text{HC}_0^{\text{per}}(\mathcal{D}_X^{\text{an}}) \rightarrow \prod_{p=-\infty}^{\infty} H^{2n-2p}(X^{\text{an}}, \mathbb{C})$ (see [EnFe], [PPT] and [Will]). Further, one has a natural map

$(-)^{an} : \mathrm{HC}_0^{per}(\mathcal{D}_X) \rightarrow \mathrm{HC}_0^{per}(\mathcal{D}_{X^{an}})^5$. The natural homomorphism $\mathcal{O}_X \rightarrow \mathcal{D}_X$ of sheaves of algebras on X induces maps on Hochschild as well as negative cyclic and periodic cyclic homologies. These maps shall be denoted by ι . The following proposition is closely related to a Theorem in [BNT] (also see [BNT1] and [BNT2]).

Proposition 3. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{HC}_0^{per}(\mathcal{O}_X) & \xrightarrow{(-)^{an} \circ \iota} & \mathrm{HC}_0^{per}(\mathcal{D}_{X^{an}}) \\ \downarrow I_{HKR} & & \downarrow \chi \\ \prod_{p=-\infty}^{\infty} \mathrm{H}^{2p}(X^{an}, \mathbb{C}) & \xrightarrow{(-\wedge \mathrm{Td}(T_X))} & \prod_{p=-\infty}^{\infty} \mathrm{H}^{2n-2p}(X^{an}, \mathbb{C}) \end{array}$$

Note that for any sheaf of algebras \mathcal{A} on X , one has natural maps $\mathrm{HC}_0^-(\mathcal{A}) \rightarrow \mathrm{HC}_0^{per}(\mathcal{A})$ and $\mathrm{HC}_0^-(\mathcal{A}) \rightarrow \mathrm{HH}_0(\mathcal{A})$. Also recall that one has a natural projection $\mathrm{H}^{2p}(X^{an}, \mathbb{C}) \rightarrow \mathrm{H}^{p,p}(X^{an}, \mathbb{C})$ for all p . We omit the proof of the following proposition.

Proposition 4. The following diagrams commute.

(a)

$$\begin{array}{ccc} \mathrm{HC}_0^-(\mathcal{O}_X) & \xrightarrow{I_{HKR}} & \prod_{p=-\infty}^{\infty} \mathrm{H}^{2p}(X^{an}, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{I_{HKR}} & \oplus_p \mathrm{H}^{p,p}(X^{an}, \mathbb{C}) \end{array}$$

(b)

$$\begin{array}{ccc} \mathrm{HC}_0^-(\mathcal{D}_{X^{an}}) & \xrightarrow{\chi} & \prod_{p=-\infty}^{\infty} \mathrm{H}^{2n-2p}(X^{an}, \mathbb{C}) \\ \downarrow & & \downarrow \\ \mathrm{HH}_0(\mathcal{D}_{X^{an}}) & \xrightarrow{\chi} & \mathrm{H}^{2n}(X^{an}, \mathbb{C}) \end{array}$$

(c)

$$\begin{array}{ccc} \mathrm{HC}_0^-(\mathcal{O}_X) & \xrightarrow{(-)^{an} \circ \iota} & \mathrm{HC}_0^-(\mathcal{D}_{X^{an}}) \\ \downarrow & & \downarrow \\ \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{an} \circ \iota} & \mathrm{HH}_0(\mathcal{D}_{X^{an}}) \end{array}$$

Proposition 5. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{an} \circ \iota} & \mathrm{HH}_0(\mathcal{D}_{X^{an}}) \\ \downarrow I_{HKR} & & \downarrow \chi \\ \oplus_p \mathrm{H}^{p,p}(X^{an}, \mathbb{C}) & \xrightarrow{(-\wedge \mathrm{Td}(T_X))_{2n}} & \mathrm{H}^{2n}(X^{an}, \mathbb{C}) \end{array}$$

Proof. We note that the natural map $\mathrm{HC}_0^-(\mathcal{O}_X) \rightarrow \mathrm{HH}_0(\mathcal{O}_X)$ is surjective. Indeed, after applying I_{HKR} , we are reduced to verifying that $\mathrm{H}^p(X, \mathrm{Ker}(d : \Omega_X^p \rightarrow \Omega_X^{p+1})) \rightarrow \mathrm{H}^p(X, \Omega_X^p)$ is surjective. By Serre's GAGA, it suffices to verify that $\mathrm{H}^p(X^{an}, \mathrm{Ker}(d : \Omega_{X^{an}}^p \rightarrow \Omega_{X^{an}}^{p+1})) \rightarrow \mathrm{H}^p(X^{an}, \Omega_{X^{an}}^p)$ is surjective. This follows from the fact that any closed (p, p) -form defines an element of $\mathrm{H}^p(X^{an}, \mathrm{Ker}(d : \Omega_{X^{an}}^p \rightarrow \Omega_{X^{an}}^{p+1}))$ as well.

Hence, any $y \in \mathrm{HH}_0(\mathcal{O}_X)$ lifts to an element $\tilde{y} \in \mathrm{HC}_0^-(\mathcal{O}_X)$. For notational brevity, we denote $\chi \circ (-)^{an}$ by χ for the rest of this proof. Now, $\chi \circ \iota(y) = (\chi \circ \iota(\tilde{y}))_{2n}$ by Proposition 4, parts (b) and (c). Further, $(\chi \circ \iota(\tilde{y}))_{2n} = (I_{HKR}(\tilde{y}) \wedge \mathrm{Td}(T_X))_{2n}$ by Proposition 3. Finally, $(I_{HKR}(\tilde{y}) \wedge \mathrm{Td}(T_X))_{2n} = (I_{HKR}(y) \wedge \mathrm{Td}(T_X))_{2n}$ by Proposition 3, part (a) and the fact that $\mathrm{Td}(T_X) \in \oplus_p \mathrm{H}^{p,p}(X^{an}, \mathbb{C})$. \square

⁵Indeed, if $f : X^{an} \rightarrow X$ is the canonical map, one has a natural map $f^{-1}(\mathcal{CC}_{\bullet}^{per}(\mathcal{D}_X)) \rightarrow \mathcal{CC}_{\bullet}^{per}(\mathcal{D}_{X^{an}})$ of complexes of sheaves on X^{an} , and hence in the derived category $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X^{an}))$ of sheaves of \mathbb{C} -vector spaces on X^{an} . By adjunction, one gets a natural map $\mathcal{CC}_{\bullet}^{per}(\mathcal{D}_X) \rightarrow Rf_*(\mathcal{CC}_{\bullet}^{per}(\mathcal{D}_{X^{an}}))$, to which we apply $R\Gamma(X, -)$. Rf_* and $R\Gamma$ are extended to $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X^{an}))$ and $\mathrm{D}(\mathrm{Sh}_{\mathbb{C}}(X))$ respectively since f_* and $\Gamma(X, -)$ have finite cohomological dimension.

The following proposition is a crucial point in this note.

Proposition 6. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{D} & \mathbb{H}^0(X, \Delta^! \Delta_* \omega_X) \\ \downarrow (-)^{an} \circ \iota & & \downarrow (\widehat{I_{HKR}}^{-1}(-))_{2n} \\ \mathrm{HH}_0(\mathcal{D}_X) & \xrightarrow{\chi} & \mathrm{H}^{2n}(X^{an}, \mathbb{C}) \end{array}$$

Proof. Let $\pi : X \rightarrow pt$ be the natural projection. The object \mathcal{O}_X of $\mathrm{Perf}(\mathcal{O}_{X \times pt})$ induces a morphism $\pi_* : \mathrm{Perf}(\mathcal{O}_X) \rightarrow \mathrm{Perf}(pt)$ in the homotopy category $\mathrm{Ho}(dg-cat)$ of DG-categories modulo quasiequivalences (see Section 8 of [T]). The notation π_* is justified by the fact that the functor from $\mathrm{D}(\mathrm{Perf}(X))$ to $\mathrm{D}(\mathrm{Perf}(pt))$ induced by π_* is indeed the derived pushforward π_* . This induces a map $\pi_* : \mathrm{HH}_0(\mathcal{O}_X) \rightarrow \mathrm{HH}_0(\mathcal{O}_{pt}) = \mathbb{C}$ which coincides with the pushforward on Hochschild homologies from [KS] (see Theorem 5 of [Ram3]). On the other hand, one has $\pi_* : \oplus_p \mathrm{H}^{p,p}(X^{an}, \mathbb{C}) \rightarrow \mathrm{H}^0(pt, \mathbb{C}) = \mathbb{C}$, which coincides with $\int_{X^{an}}$. By the proof of Proposition 5.2.3 of [KS], $\widehat{I_{HKR}}^{-1} \circ D$ commutes with π_* . On the other hand, let $\mathrm{Perf}(\mathcal{D}_X)$ denote the DG-category of perfect complexes of (right) \mathcal{D}_X -modules that are quasi-coherent as \mathcal{O}_X -modules. One has a map $\pi_*^{\mathcal{D}} : \mathrm{Perf}(\mathcal{D}_X) \rightarrow \mathrm{Perf}(pt)$ in $\mathrm{Ho}(dg-cat)$. The functor induced by $\pi_*^{\mathcal{D}}$ on derived categories maps $M \in \mathrm{D}(\mathrm{Perf}(\mathcal{D}_X))$ to $\pi_*(M^{an} \otimes_{\mathcal{D}_{X^{an}}}^{\mathbb{L}} \mathcal{O}_{X^{an}})^6$. By Section 8 of [T], $\pi_*^{\mathcal{D}}$ induces a map $\pi_*^{\mathcal{D}} : \mathrm{HH}_0(\mathrm{Perf}(\mathcal{D}_X)) \rightarrow \mathrm{HH}_0(pt) \cong \mathbb{C}$ on Hochschild homologies. By Proposition 7 at the end of this section, the composite map

$$(4) \quad \mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X)) \rightarrow \mathrm{HH}_\bullet(\mathcal{D}_X) \xrightarrow{(-)^{an}} \mathrm{HH}_\bullet(\mathcal{D}_{X^{an}})$$

is an isomorphism (the first map in the above composition is the trace map from Section 4 of [K]). $\pi_*^{\mathcal{D}}$ therefore, induces a \mathbb{C} -linear functional on $\mathrm{HH}_0(\mathcal{D}_{X^{an}})$, which we shall continue to denote by $\pi_*^{\mathcal{D}}$. It follows from [EnFe] and [Ram2] that

$$\pi_*^{\mathcal{D}} = \int_{X^{an}} \circ \chi : \mathrm{HH}_0(\mathcal{D}_{X^{an}}) \rightarrow \mathbb{C}.$$

Since $\int_{X^{an}} : \mathrm{H}^{2n}(X^{an}, \mathbb{C}) \rightarrow \mathbb{C}$ is an isomorphism, the required proposition follows once we check that $\pi_* = \pi_*^{\mathcal{D}} \circ (-)^{an} \circ \iota$. This follows from the fact that the diagram

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{an} \circ \iota} & \mathrm{HH}_0(\mathcal{D}_X) \\ \uparrow & & \uparrow \\ \mathrm{HH}_0(\mathrm{Perf}(\mathcal{O}_X)) & \xrightarrow{(-) \otimes_{\mathcal{O}_X} \mathcal{D}_X} & \mathrm{HH}_0(\mathrm{Perf}(\mathcal{D}_X)) \end{array}$$

(the left vertical arrows being the trace isomorphism from Section 4 of [K] and the right vertical arrow being the composite map (4)) commutes as well as the observation that for $E \in \mathrm{D}(\mathrm{Perf}(\mathcal{O}_X))$,

$$\pi_*^{\mathcal{D}} \iota(E) = \pi_*((E \otimes_{\mathcal{O}_X} \mathcal{D}_X)^{an} \otimes_{\mathcal{D}_{X^{an}}}^{\mathbb{L}} \mathcal{O}_{X^{an}}) = \pi_* E^{an}$$

(recall that $\pi_* E = \pi_* E^{an}$ in $\mathrm{D}(\mathrm{Perf}(pt))$ by Serre's GAGA). \square

By Propositions 5 and 6

$$(I_{HKR}(\alpha) \wedge \tau(1))_{2n} = (I_{HKR}(\alpha) \wedge \mathrm{Td}(T_X))_{2n}$$

for all $\alpha \in \mathrm{HH}_0(\mathcal{O}_X)$. Hence, $\mathrm{eu}(\mathcal{O}_X) = \tau(1) = \mathrm{Td}(T_X)$. To complete the proof of Proposition 5, we sketch the proof of the following proposition.

Proposition 7. $\mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X)) \cong \mathrm{HH}_\bullet(\mathcal{D}_{X^{an}})$. This isomorphism is realized by the composite map (4).

Proof. One has to verify that the arguments of B. Keller in Section 5 of [K] go through when \mathcal{O}_X is replaced by \mathcal{D}_X . The crucial part here is the analog of Theorem 5.5 of [K] (originally proven as Propositions 5.2.2-5.2.4 of [TT]) when \mathcal{O}_X is replaced by \mathcal{D}_X . This is done in Propositions 3.3.1-3.3.3 of [DY] (which prove the analog of Theorem 5.5 of [K] in a much more general setting: in particular, when \mathcal{O}_X is replaced by \mathcal{R}_X where \mathcal{R}_X is a sheaf of quasicoherent \mathcal{O}_X -algebras (possibly noncommutative)). Let Y be any quasi-compact,

⁶The latter is indeed in $\mathrm{D}(\mathrm{Perf}(pt))$: see [ScS] for instance.

quasi-separated scheme over \mathbb{C} with V, W quasi-compact open subschemes of Y such that $Y = V \cup W$. Following the arguments of Sections 5.6 and 5.7 of [K], one obtains a morphism of Mayer-Vietoris sequences

$$\begin{array}{ccccccc} \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_Y)) & \longrightarrow & \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_V)) \oplus \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_W)) & \longrightarrow & \mathrm{HH}_i(\mathrm{Perf}(\mathcal{D}_{V \cap W})) & \longrightarrow & \mathrm{HH}_{i-1}(\mathrm{Perf}(\mathcal{D}_Y)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{HH}_i(\mathcal{D}_{Y^{an}}) & \longrightarrow & \mathrm{HH}_i(\mathcal{D}_{V^{an}}) \oplus \mathrm{HH}_i(\mathcal{D}_{W^{an}}) & \longrightarrow & \mathrm{HH}_i(\mathcal{D}_{(V \cap W)^{an}}) & \longrightarrow & \mathrm{HH}_{i-1}(\mathcal{D}_{Y^{an}}) \end{array}$$

(for each $i \in \mathbb{Z}$). The vertical arrows in the above diagram are induced by the composite map (4). As in Section 5.9 of [K], we may then reduce the proof of the desired proposition to proving the desired proposition when X is affine with trivial tangent bundle. For the rest of this proof, we assume that this is indeed the case.

Since $\mathcal{D}_X - \mathrm{mod}$ denote the Abelian category of (right) \mathcal{D}_X -modules are quasi-coherent \mathcal{O}_X -modules. There is an equivalence of abelian categories between $D_X - \mathrm{mod}$ and $\mathcal{D}_X - \mathrm{mod}$, where $D_X := \Gamma(X, \mathcal{D}_X)$ (see [DY], example 1.1.5). Hence, one has an equivalence of DG-categories between $\mathrm{Perf}(D_X)$ and $\mathrm{Perf}(\mathcal{D}_X)$ (this follows, for instance, from Lemma 2.2.1 of [DY]). This equivalence induces an isomorphism $\mathrm{HH}_\bullet(\mathrm{Perf}(D_X)) \xrightarrow{\cong} \mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X))$. Further, there is a natural map $\mathrm{HH}_\bullet(D_X) \rightarrow \mathrm{HH}_\bullet(\mathcal{D}_X)$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{HH}_\bullet(\mathrm{Perf}(D_X)) & \xrightarrow{\cong} & \mathrm{HH}_\bullet(\mathrm{Perf}(\mathcal{D}_X)) \\ \downarrow \cong & & \downarrow \\ \mathrm{HH}_\bullet(D_X) & \longrightarrow & \mathrm{HH}_\bullet(\mathcal{D}_X) \end{array}$$

In the above diagram, the vertical arrows are trace maps from Section 4 of [K]. For honest algebras, they yield isomorphisms. We are therefore, reduced to verifying that the composite map

$$(5) \quad \mathrm{HH}_\bullet(D_X) \rightarrow \mathrm{HH}_\bullet(\mathcal{D}_X) \xrightarrow{(-)^{an}} \mathrm{HH}_\bullet(\mathcal{D}_{X^{an}})$$

is an isomorphism. Let $\mathcal{D}_{X^{an}}^\bullet$ denote the Dolbeault resolution of the sheaf $\mathcal{D}_{X^{an}}$. This is a sheaf of DG-algebras on X . Let $C_\bullet(\mathcal{D}_{X^{an}}^\bullet)$ denote the complex of global sections of the complex of completed Hochschild chains on X (see [Ram2], Section 3.3). There is a natural map of complexes $C_\bullet(D_X) \rightarrow C_\bullet(\mathcal{D}_{X^{an}}^\bullet)$ inducing (5) on homology. To prove that this is a quasi-isomorphism, we filter algebraic and holomorphic differential operators by order and consider the induced map on the E^2 -terms of the spectral sequences from Section 3.3 of [Bryl]. This turns out to be induced on homology by the natural map from the algebraic De-Rham complex $(\Omega^{2n-\bullet}(T^*X), d_{DR}^{alg})$ to the Dolbeault complex $(\Gamma(X^{an}, \Omega_{T^*X^{an}}^{2n-\bullet} \otimes_{\mathcal{O}_{X^{an}}} \Omega_{X^{an}}^{0,\bullet}), d + \bar{\partial})$ ⁷. That this is a quasiisomorphism amounts to the assertion that natural map from the algebraic De-Rham complex of X to the smooth De-Rham complex of X^{an} is a quasiisomorphism (see [Groth]).

□

4. A PROOF OF PROPOSITION 3.

One notes that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{HH}_0(\mathcal{O}_X) & \xrightarrow{(-)^{an}} & \mathrm{HH}_0(\mathcal{O}_{X^{an}}) \\ \downarrow \iota & & \downarrow \iota \\ \mathrm{HH}_0(\mathcal{D}_X) & \xrightarrow{(-)^{an}} & \mathrm{HH}_0(\mathcal{D}_{X^{an}}) \end{array}$$

⁷Here, $\Omega_{T^*X^{an}}^\bullet$ is the complex of sheaves on X^{an} whose sections on each open subset U of X^{an} are holomorphic forms on T^*U that are algebraic along the fibres of the projection $T^*U \rightarrow U$. d is the (holomorphic) De-Rham differential on this complex.

To prove Proposition 3, it therefore, suffices to show that the following diagram commutes (where $Y := X^{an}$).

$$(6) \quad \begin{array}{ccc} \mathrm{HC}_0^{per}(\mathcal{O}_Y) & \xrightarrow{\iota} & \mathrm{HC}_0^{per}(\mathcal{D}_Y) \\ \downarrow I_{HKR} & & \downarrow \chi \\ \prod_{p=-\infty}^{\infty} H^{2p}(Y, \mathbb{C}) & \xrightarrow{(-\wedge \mathrm{Td}(T_Y))} & \prod_{p=-\infty}^{\infty} H^{2n-2p}(Y, \mathbb{C}) \end{array}$$

In other words, we now work with a complex manifold rather than an algebraic variety. Recall that there is a deformation quantization $\mathbb{A}_{T^*Y}^{\hbar}$ of $\mathcal{O}_{T^*Y}[[\hbar]]$ such that $\pi^{-1}\mathcal{D}_Y \hookrightarrow \mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]$ and $\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]$ is flat over $\pi^{-1}\mathcal{D}_Y$. Here, $\pi : T^*Y \rightarrow Y$ is the canonical projection.

In this situation, one has a natural map $\pi^{-1} : \mathrm{HC}_0^{per}(\mathcal{D}_Y) \rightarrow \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}])$. Indeed, if $\mathcal{U} := \{U_i\}$ is a good open cover of Y , one has a natural map of complexes between the periodic cyclic-Cech complex $C^{\vee}(\mathcal{U}, \mathcal{CC}_{\bullet}^{per}(\mathcal{D}_Y))$ and $C^{\vee}(\mathcal{V}, \mathcal{CC}_{\bullet}^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]))$ where $\mathcal{V} := \{\pi^{-1}(U_i)\}$. Similarly, one has a natural map $\pi^{-1} : \mathrm{HC}_0^{per}(\mathcal{O}_Y) \rightarrow \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar})$. Further, one has a trace density map $\chi_{FFS} : \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]) \rightarrow \prod_p H^{2n-2p}(T^*Y, \mathbb{C})((\hbar))$ (see [BNT], [EnFe], [FFS], [Will]). Note that we can compose χ_{FFS} with the natural map $\beta : \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}) \rightarrow \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}])$. We shall abuse notation to denote $\chi_{FFS} \circ \beta$ by χ_{FFS} . Let $i : Y \rightarrow T^*Y$ denote inclusion as the zero section. The following proposition is clear.

Proposition 8. The diagram

$$\begin{array}{ccc} \mathrm{HC}_0^{per}(\mathcal{D}_Y) & \xrightarrow{\pi^{-1}} & \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]) \\ \downarrow \chi & & \downarrow \chi_{FFS} \\ \prod_p H^{2n-2p}(Y, \mathbb{C})((\hbar)) & \xrightarrow{\pi^*} & \prod_p H^{2n-2p}(T^*Y, \mathbb{C})((\hbar)) \end{array}$$

commutes. Further, $i^* \circ \pi^* = \mathrm{id}$ on $\prod_p H^{2n-2p}(Y, \mathbb{C})((\hbar))$.

One has a “principal symbol” homomorphism $\sigma : \mathbb{A}_{T^*Y}^{\hbar} \rightarrow \mathcal{O}_{T^*Y}$. The following theorem is from [BNT] (see also [BNT1] and [BNT2]).

Theorem 2. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}) & \xrightarrow{\sigma} & \mathrm{HC}_0^{per}(\mathcal{O}_{T^*Y}) \\ \downarrow \chi_{FFS} & & \downarrow I_{HKR} \\ \prod_p H^{2n-2p}(T^*Y, \mathbb{C})((\hbar)) & \xleftarrow{(-) \cup \pi^* \mathrm{Td}(T_Y)} & \prod_p H^{2p}(T^*Y, \mathbb{C})((\hbar)) \end{array}$$

Proposition 9. The following diagrams commute.

$$\begin{array}{ccc} \mathrm{HC}_0^{per}(\mathcal{O}_Y) & \xrightarrow{\pi^{-1}} & \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}) \\ \downarrow \pi^* & & \downarrow \mathrm{id} \\ \mathrm{HC}_0^{per}(\mathcal{O}_{T^*Y}) & \xleftarrow{\sigma} & \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}) \\ \mathrm{HC}_0^{per}(\mathcal{O}_Y) & \xrightarrow{\iota} & \mathrm{HC}_0^{per}(\mathcal{D}_Y) \\ \downarrow \pi^{-1} & & \downarrow \pi^{-1} \\ \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}) & \xrightarrow{\beta} & \mathrm{HC}_0^{per}(\mathbb{A}_{T^*Y}^{\hbar}[\hbar^{-1}]) \\ \mathrm{HC}_0^{per}(\mathcal{O}_Y) & \xrightarrow{\pi^*} & \mathrm{HC}_0^{per}(\mathcal{O}_{T^*Y}) \\ \downarrow I_{HKR} & & \downarrow I_{HKR} \\ \prod_p H^{2p}(Y, \mathbb{C}) & \xrightarrow{\pi^*} & \prod_p H^{2p}(T^*Y, \mathbb{C})((\hbar)) \end{array}$$

Denote the bottom arrow in the diagram of equation (6) by θ (after extending scalars to $\mathbb{C}((\hbar))$ in the codomain). Since $I_{HKR} : \mathrm{HC}_0^{\mathrm{per}}(\mathcal{O}_Y) \rightarrow \prod_p H^{2p}(Y, \mathbb{C})$ is an isomorphism, Propositions 8,9 and Theorem 2 together imply that

$$\theta(\alpha) = i^*(\pi^*(\alpha) \cup \pi^*(\mathrm{Td}(T_Y))) = i^*\pi^*(\alpha \cup \mathrm{Td}(T_Y)) = \alpha \cup \mathrm{Td}(T_Y).$$

This proves Proposition 3.

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